BI-ORTHOGONAL SYSTEMS ON THE UNIT CIRCLE, REGULAR SEMI-CLASSICAL WEIGHTS AND THE DISCRETE GARNIER EQUATIONS

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Abstract. We demonstrate that a system of bi-orthogonal polynomials and their associated functions corresponding to a regular semi-classical weight on the unit circle constitute a class of general classical solutions to the Garnier systems by explicitly constructing its Hamiltonian formulation and showing that it coincides with that of a Garnier system. Such systems can also be characterised by recurrence relations of the discrete Painlevé type, for example in the case with one free deformation variable the system was found to be characterised by a solution to the discrete fifth Painlevé equation. Here we derive the canonical forms of the multi-variable generalisation of the discrete fifth Painlevé equation to the Garnier systems, i.e. for arbitrary numbers of deformation variables.

1. General Structures of Bi-orthogonality

Consider a formal complex weight $w(z)$ and its Fourier coefficients $\{w_k\}_{k\in\mathbb{Z}}$ defined by

$$w(z) = \sum_{k=-\infty}^{\infty} w_k z^k, \quad w_k = \int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta)\zeta^{-k}, \tag{1.1}$$

with support on the unit circle where $\mathbb{T}$ denotes the unit circle $|\zeta| = 1$ with $\zeta = e^{i\theta}$, $\theta \in [0, 2\pi)$. The Toeplitz determinants constructed from the these Fourier coefficients are related to averages over the unitary group $U(n)$ with respect to the Haar measure by the Heine formula[10],[11]

$$\begin{aligned} I_n[w] := \Big\langle \prod_{l=1}^{n} w(z_l) \Big\rangle_{U(n)} &= \frac{1}{(2\pi)^n n!} \int_0^{2\pi} d\theta_1 \cdots \int_0^{2\pi} d\theta_n \prod_{l=1}^{n} w(e^{i\theta_l}) \prod_{1\le j<k\le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \\ &= \det[w_{i-j}]_{i,j=0,\dots,n-1}, \quad n \ge 1, \quad I_0[w] = 1. \end{aligned} \tag{1.2}$$

Note that $\bar{w}_n \neq w_{-n}$ in general and consequently the Toeplitz matrix $(w_{i-j})_{i,j=0,\dots,n-1}$ is not necessarily Hermitian.

Notwithstanding the fact that many physically interesting quantities are characterised as averages over the unitary group $U(n)$ we wish to emphasise another perspective. Intimately connected with such averages are systems of bi-orthogonal polynomials on the unit circle which are orthogonal with respect to the weight $w(z)$ underlying the $U(n)$ average, in the sense of (1.4) and (1.5). Such systems are equivalent to systems of bi-orthogonal Laurent polynomials, first introduced by Jones and Thron [24] and studied subsequently by [25] amongst others, as was shown by Hendriksen and van Rossum [17] and Pastro [32] so that all of our conclusions apply equally well to these systems. A particular class of weights of great interest is the generic or regular semi-classical class which are parameterised by the co-ordinates and residues of singular points $\{z_j\}_{j=1}^{M}$ and $\{\rho_j\}_{j=1}^{M}$ respectively (see (2.1) for the definition). Some of the relevant properties of this class are summarised in a self-contained way in Section 2. The important fact that is relevant here is that systems of bi-orthogonal polynomials and their associated functions with such weights have the property that their monodromy in the complex spectral variable z is preserved under arbitrary deformations of the singularity co-ordinates $\{z_j\}_{j=1}^{M}$. This fact was first derived in the context of bi-orthogonal polynomial systems on the unit circle in [8], although it was known to be true for systems of orthogonal polynomials on the line due to work by Magnus [29]. This later result was subsequently extended to orthogonal polynomial systems with a certain type of non-generic or degenerate semi-classical weight by Bertola, Eynard and Harnad

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[5]. The isomonodromic character of bi-orthogonal Laurent polynomial systems with such weights has also been established independently by Bertola and Gekhtman [6]. Here we specifically demonstrate that bi-orthogonal polynomial systems on the unit circle with regular semi-classical weights constitute a class of general classical solutions to the Garnier systems. We carry out this task in Proposition 3.1 by proving that the dynamical equations of the system with respect to the singularity co-ordinates $\{z_j\}_{j=1}^{M}$ are Hamiltonian and that this coincides precisely with the Garnier system $\mathcal{G}_N \equiv \{q_j, p_j; K_j, z_j\}$. The reader should note that the term classical refers to two distinct notions: semi-classical orthogonal polynomial systems are ones which generalise the classical systems such as the Hermite, Laguerre, Jacobi or any member of the Askey Table, whilst a classical solution of a Painlevé or Garnier equation is a special solution constructible from hypergeometric functions and implies a condition on the parameters and boundary/initial conditions.

The significance of this observation is that one can reverse the usual argument and use the structures derived from approximation theory to deduce new results about the integrable system. One particular consequence of the identification of the $U(n)$ averages with the Garnier system is their characterisation by recurrence relations of the discrete Painlevé type. In the case of the simplest $U(n)$ average with one free deformation variable the system was found to be characterised by a solution to the discrete fifth Painlevé equation, see Proposition 4.1 in Section 4. In this section we derive the canonical forms of the higher analogues of the discrete fifth Painlevé for the Garnier systems, i.e. for the many deformation variable case from the approximation theory structures. We give the explicit coupled recurrence relations for the two variable Garnier system in Proposition 4.2, and the arbitrary variable recurrence relations in Proposition 4.3, and these constitute our key results.

We define bi-orthogonal polynomials $\{\phi_n(z), \bar{\phi}_n(z)\}_{n=0}^{\infty}$ with respect to the weight $w(z)$ on the unit circle by the orthogonality relation

$$\int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta)\phi_m(\zeta)\bar{\phi}_n(\bar{\zeta}) = \delta_{m,n}, \quad m, n \in \mathbb{Z}_{\geq 0}. \tag{1.3}$$

Alternatively one can express this definition in terms of orthogonality with respect to the monomial basis

$$\int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta)\phi_n(\zeta)\bar{\zeta}^m = \begin{cases} 0 & m < n \\ 1/\kappa_n & m = n \end{cases}, \tag{1.4}$$

$$\int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta)\zeta^m\bar{\phi}_n(\bar{\zeta}) = \begin{cases} 0 & m < n \\ 1/\kappa_n & m = n \end{cases}. \tag{1.5}$$

Notwithstanding the notation, $\bar{\phi}_n$ is not in general equal to the complex conjugate of ϕ_n. The leading coefficients of the polynomials are specified by

$$\frac{\phi_n(z)}{\kappa_n} = z^n + \lambda_n z^{n-1} + \mu_n z^{n-2} + \ldots + r_n, \tag{1.6}$$

$$\frac{\bar{\phi}_n(z)}{\kappa_n} = z^n + \bar{\lambda}_n z^{n-1} + \bar{\mu}_n z^{n-2} + \ldots + \bar{r}_n, \tag{1.7}$$

where again $\bar{\lambda}_n$, $\bar{\mu}_n$, $\bar{r}_n$ are not in general equal to the corresponding complex conjugates of λ_n, μ_n, r_n respectively. We define a double sequence of r-coefficients by

$$r_n = \frac{\phi_n(0)}{\kappa_n}, \quad \bar{r}_n = \frac{\bar{\phi}_n(0)}{\kappa_n}, \quad n \geq 1, \quad r_0 = \bar{r}_0 = 1, \tag{1.8}$$

which differ slightly from the standard definitions of the reflection or Verblunsky coefficients α_n, in that $\alpha_n = -\bar{r}_{n+1}$. The polynomial coefficients introduced above are related by a system of coupled recurrence equations, the first two being [10]

$$\kappa_n^2 - \kappa_{n-1}^2 = \phi_n(0)\bar{\phi}_n(0), \quad \lambda_n - \lambda_{n-1} = r_n\bar{r}_{n-1}. \tag{1.9}$$

We have a extension of the standard results on the existence of orthogonal polynomial systems on the unit circle to the bi-orthogonal setting due to Baxter.

Proposition 1.1 ([3]). *The bi-orthogonal system $\{\phi_n, \bar{\phi}_n\}_{n=0}^{\infty}$ exists if and only if $I_n \neq 0$ for all $n \in \mathbb{N}$.*

It is a well known result in the theory of Toeplitz determinants [34] that

$$\frac{I_{n+1}[w]I_{n-1}[w]}{(I_n[w])^2} = 1 - r_n\bar{r}_n, \quad n \geq 1. \tag{1.10}$$

Rather than dealing with $\bar{\phi}_n$ we prefer to use the reciprocal polynomial $\phi_n^*(z)$ defined in terms of the nth degree polynomial $\bar{\phi}_n(z)$ by

$$\phi_n^*(z) := z^n\bar{\phi}_n(1/z). \tag{1.11}$$

The generating function of the Toeplitz elements, known as the Carathéodory function

$$F(z) := \int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta}\frac{\zeta+z}{\zeta-z}w(\zeta), \tag{1.12}$$

will also feature prominently in our work. The fundamental object identified in the study [8] is the 2×2 matrix

$$Y_n(z;t) := \begin{pmatrix} \phi_n(z) & \epsilon_n(z)/w(z) \\ \phi_n^*(z) & -\epsilon_n^*(z)/w(z) \end{pmatrix}, \tag{1.13}$$

where the associated functions or functions of the second kind are defined by

$$\epsilon_n(z) := \int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta}\frac{\zeta+z}{\zeta-z}w(\zeta)\phi_n(\zeta), \quad n \geq 1, \quad \epsilon_0(z) = \kappa_0[w_0 + F(z)], \tag{1.14}$$

$$\epsilon_n^*(z) := -z^n\int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta}\frac{\zeta+z}{\zeta-z}w(\zeta)\bar{\phi}_n(\bar{\zeta}), \quad n \geq 1, \quad \epsilon_0^*(z) = \kappa_0[w_0 - F(z)]. \tag{1.15}$$

Solutions to the orthogonality relations yield the following determinantal and integral representations for the polynomials,

$$\phi_n(z) = \frac{\kappa_n}{I_n}\det\begin{pmatrix} w_0 & \dots & w_{-j} & \dots & w_{-n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{n-1} & \dots & w_{n-j-1} & \dots & w_{-1} \\ 1 & \dots & z^j & \dots & z^n \end{pmatrix} = (-1)^n\kappa_n\frac{I_n[w(\zeta)(\zeta-z)]}{I_n[w(\zeta)]}, \tag{1.16}$$

$$\phi_n^*(z) = \frac{\kappa_n}{I_n}\det\begin{pmatrix} w_0 & \dots & w_{-n+1} & z^n \\ \vdots & \vdots & \vdots & \vdots \\ w_{n-j} & \dots & w_{-j+1} & z^j \\ \vdots & \vdots & \vdots & \vdots \\ w_n & \dots & w_1 & 1 \end{pmatrix} = \kappa_n\frac{I_n[w(\zeta)(1-z\zeta^{-1})]}{I_n[w(\zeta)]}. \tag{1.17}$$

The associated functions have representations analogous to (1.16,1.17)

$$\frac{\kappa_n}{2}\epsilon_n(z) = \frac{1}{2I_{n+1}}\det\begin{pmatrix} w_0 & \dots & w_{-j} & \dots & w_{-n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ w_{n-1} & \dots & w_{n-j-1} & \dots & w_{-1} \\ g_0 & \dots & g_j & \dots & g_n \end{pmatrix} = z^n\frac{I_{n+1}[w(\zeta)(1-z\zeta^{-1})^{-1}]}{I_{n+1}[w(\zeta)]}, \tag{1.18}$$

$$\frac{\kappa_n}{2}\epsilon_n^*(z) = (-1)^{n+1}\frac{1}{2I_{n+1}}\det\begin{pmatrix} w_0 & \dots & w_{-n+1} & g_n \\ \vdots & \vdots & \vdots & \vdots \\ w_{n-j} & \dots & w_{-j+1} & g_j \\ \vdots & \vdots & \vdots & \vdots \\ w_n & \dots & w_1 & g_0 \end{pmatrix} = (-z)^{n+1}\frac{I_{n+1}[w(\zeta)(\zeta-z)^{-1}]}{I_{n+1}[w(\zeta)]}. \tag{1.19}$$

where

$$g_n(z) := 2z\int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta}w(\zeta)\frac{\zeta^n}{\zeta-z}, \quad n \geq 0, \tag{1.20}$$

for $|z| \neq 1$.

From the general properties of bi-orthogonality we can deduce that the matrix Y_n obeys a difference system.

Proposition 1.2 ([10]). *Assuming $\kappa_n \neq 0$ (or equivalently $I_n \neq 0$) for $n \in \mathbb{N}$ the matrix Y_n satisfies the recurrence relation in n*

$$Y_{n+1} := K_n Y_n = \frac{1}{\kappa_n}\begin{pmatrix} \kappa_{n+1} z & \phi_{n+1}(0) \\ \bar{\phi}_{n+1}(0) z & \kappa_{n+1} \end{pmatrix} Y_n. \tag{1.21}$$

Theorem 1.1 ([10]). *The Casoratians of the solutions $\phi_n, \phi^*_n, \epsilon_n, \epsilon^*_n$ to the above recurrence relations are*

$$\phi_{n+1}(z)\epsilon_n(z) - \epsilon_{n+1}(z)\phi_n(z) = 2\frac{\phi_{n+1}(0)}{\kappa_n} z^n, \tag{1.22}$$

$$\phi^*_{n+1}(z)\epsilon^*_n(z) - \epsilon^*_{n+1}(z)\phi^*_n(z) = 2\frac{\bar{\phi}_{n+1}(0)}{\kappa_n} z^{n+1}, \tag{1.23}$$

$$\phi_n(z)\epsilon^*_n(z) + \epsilon_n(z)\phi^*_n(z) = 2z^n. \tag{1.24}$$

Under quite general conditions the matrix system Y_n obeys the following spectral differential system.

Proposition 1.3 ([19],[8]). *Assume that the weight satisfies the moment conditions*

$$\int_{\mathbb{T}} \frac{d\zeta}{2\pi i\zeta} w(\zeta) \frac{\frac{d}{dz}\log w(z) - \frac{d}{d\zeta}\log w(\zeta)}{z-\zeta} \zeta^j \neq \infty, \quad j \in \mathbb{Z}. \tag{1.25}$$

Then the matrix Y_n satisfies the differential relation with respect to the spectral variable z

$$\frac{d}{dz} Y_n := A_n Y_n = \frac{1}{W(z)} \begin{pmatrix} -\left[\Omega_n(z) + V(z) - \frac{\kappa_{n+1}}{\kappa_n} z\Theta_n(z)\right] & \frac{\phi_{n+1}(0)}{\kappa_n}\Theta_n(z) \\ -\frac{\bar{\phi}_{n+1}(0)}{\kappa_n} z\Theta^*_n(z) & \Omega^*_n(z) - V(z) - \frac{\kappa_{n+1}}{\kappa_n}\Theta^*_n(z) \end{pmatrix} Y_n. \tag{1.26}$$

The utility of such a parameterisation of the spectral matrix A_n will be evident when we make the specialisation to the regular semi-classical weights. We will refer to $\Theta_n, \Omega_n, \Theta^*_n, \Omega^*_n$ as spectral coefficients.

The scalar differential equation system corresponding to the above matrix system is specified in the following result.

Proposition 1.4. *The components of the matrix Y_n satisfy two second-order scalar ordinary differential equations in the spectral variable: $\phi_n(z)$ or $\epsilon_n(z)/w(z)$ satisfy*

$$\phi''_n + p_1\phi'_n + p_2\phi_n = 0, \tag{1.27}$$

while $\phi^*_n(z)$ and $-\epsilon^*_n(z)/w(z)$ satisfy

$$\phi^{*''}_n + p^*_1\phi^{*'}_n + p^*_2\phi^*_n = 0. \tag{1.28}$$

The coefficients of the scalar second-order differential equations are

$$p_1(z) = \frac{W'}{W} - \frac{\Theta'_n}{\Theta_n} + \frac{2V}{W} - \frac{n}{z}, \tag{1.29}$$

and

$$\begin{aligned} p_2(z) = {} & \frac{\Theta_n(\Omega'_n + V') - \Theta'_n(\Omega_n + V)}{W\Theta_n} - \frac{\kappa_{n+1}}{\kappa_n}\frac{\Theta_n}{W} \\ & - \frac{\left[\Omega_n + V - \frac{\kappa_{n+1}}{\kappa_n} z\Theta_n\right]\left[\Omega^*_n - V - \frac{\kappa_{n+1}}{\kappa_n}\Theta^*_n\right]}{W^2} + \frac{\phi_{n+1}(0)\bar{\phi}_{n+1}(0) z\Theta_n\Theta^*_n}{\kappa_n^2 W^2}, \end{aligned} \tag{1.30}$$

and

$$p^*_1(z) = \frac{W'}{W} - \frac{\Theta^{*'}_n}{\Theta^*_n} + \frac{2V}{W} - \frac{n+1}{z}, \tag{1.31}$$

and

$$
(1.32)\quad p_2^*(z) = \frac{(z^{-1}\Theta_n^* + \Theta_n^{*\prime})(\Omega_n^* - V) - \Theta_n^*(\Omega_n^{*\prime} - V')}{W\Theta_n^*} - \frac{\kappa_{n+1}}{\kappa_n}\frac{\Theta_n^*}{zW} - \frac{\left[\Omega_n + V - \frac{\kappa_{n+1}}{\kappa_n}z\Theta_n\right]\left[\Omega_n^* - V - \frac{\kappa_{n+1}}{\kappa_n}\Theta_n^*\right]}{W^2} + \frac{\phi_{n+1}(0)\bar{\phi}_{n+1}(0)z\Theta_n\Theta_n^*}{\kappa_n^2 W^2}.
$$

Proof. The two ordinary differential equations (1.27) and (1.28) follow from the elimination of ϕ_n^* and ϕ_n in (1.26) respectively. □

A consequence of the compatibility between the differential relations (1.26) and the recurrence relations (1.21) is the following collection of recurrence relations for the spectral coefficients $\Omega_n, \Omega_n^*, \Theta_n, \Theta_n^*$.

Proposition 1.5 ([8]). *Given the conditions of Proposition 1.1 the spectral coefficients* $\{\Omega_n(z), \Omega_n^*(z), \Theta_n(z), \Theta_n^*(z)\}_{n=0}^{\infty}$ *satisfy the following recurrence relations in n*

$$
(1.33)\quad \Omega_n(z) + \Omega_{n-1}(z) - \left(\frac{\phi_{n+1}(0)}{\phi_n(0)} + \frac{\kappa_{n+1}}{\kappa_n}z\right)\Theta_n(z) + (n-1)\frac{W(z)}{z} = 0,
$$

$$
(1.34)\quad \left(\frac{\phi_{n+1}(0)}{\phi_n(0)} + \frac{\kappa_{n+1}}{\kappa_n}z\right)(\Omega_{n-1}(z) - \Omega_n(z)) + \frac{\kappa_n\phi_{n+2}(0)}{\kappa_{n+1}\phi_{n+1}(0)}z\Theta_{n+1}(z) - \frac{\kappa_{n-1}\phi_{n+1}(0)}{\kappa_n\phi_n(0)}z\Theta_{n-1}(z) - \frac{\phi_{n+1}(0)}{\phi_n(0)}\frac{W(z)}{z} = 0,
$$

$$
(1.35)\quad \Omega_n^*(z) + \Omega_{n-1}^*(z) - \left(\frac{\kappa_{n+1}}{\kappa_n} + \frac{\bar{\phi}_{n+1}(0)}{\bar{\phi}_n(0)}z\right)\Theta_n^*(z) - n\frac{W(z)}{z} = 0,
$$

$$
(1.36)\quad \left(\frac{\kappa_{n+1}}{\kappa_n} + \frac{\bar{\phi}_{n+1}(0)}{\bar{\phi}_n(0)}z\right)(\Omega_{n-1}^*(z) - \Omega_n^*(z)) + \frac{\kappa_n\bar{\phi}_{n+2}(0)}{\kappa_{n+1}\bar{\phi}_{n+1}(0)}z\Theta_{n+1}^*(z) - \frac{\kappa_{n-1}\bar{\phi}_{n+1}(0)}{\kappa_n\bar{\phi}_n(0)}z\Theta_{n-1}^*(z) + \frac{\kappa_{n+1}}{\kappa_n}\frac{W(z)}{z} = 0,
$$

$$
(1.37)\quad \Omega_{n+1}(z) + \Omega_n^*(z) - \left(\frac{\phi_{n+2}(0)}{\phi_{n+1}(0)} + \frac{\kappa_{n+2}}{\kappa_{n+1}}z\right)\Theta_{n+1}(z) + \frac{\kappa_{n+1}}{\kappa_n}(z\Theta_n(z) - \Theta_n^*(z)) = 0,
$$

$$
(1.38)\quad \Omega_n(z) - \Omega_{n+1}(z) + \frac{\kappa_{n+2}}{\kappa_{n+1}}\left(z + \frac{\bar{\phi}_{n+1}(0)}{\kappa_{n+1}}\frac{\phi_{n+2}(0)}{\kappa_{n+2}}\right)\Theta_{n+1}(z) + \frac{\phi_{n+1}(0)\bar{\phi}_{n+1}(0)}{\kappa_{n+1}\kappa_n}\Theta_n^*(z) - \frac{\kappa_{n+1}}{\kappa_n}z\Theta_n(z) - \frac{W(z)}{z} = 0,
$$

$$
(1.39)\quad \Omega_{n+1}^*(z) + \Omega_n(z) - \left(\frac{\kappa_{n+2}}{\kappa_{n+1}} + \frac{\bar{\phi}_{n+2}(0)}{\bar{\phi}_{n+1}(0)}z\right)\Theta_{n+1}^*(z) - \frac{\kappa_{n+1}}{\kappa_n}(z\Theta_n(z) - \Theta_n^*(z)) - \frac{W(z)}{z} = 0,
$$

$$
(1.40)\quad \Omega_n^*(z) - \Omega_{n+1}^*(z) + \frac{\kappa_{n+2}}{\kappa_{n+1}}\left(1 + \frac{\phi_{n+1}(0)}{\kappa_{n+1}}\frac{\bar{\phi}_{n+2}(0)}{\kappa_{n+2}}z\right)\Theta_{n+1}^*(z) + \frac{\phi_{n+1}(0)\bar{\phi}_{n+1}(0)}{\kappa_{n+1}\kappa_n}z\Theta_n(z) - \frac{\kappa_{n+1}}{\kappa_n}\Theta_n^*(z) = 0.
$$

The spectral coefficients Θ_n, Ω_n are related to their "conjugates" Θ_n^*, Ω_n^* via a number of functional (or recurrence) relations so that one can characterise the system in terms of either set. We will refer to these as transition relations.

Corollary 1.1 ([8]). *The spectral coefficients are inter-related through the following equations*

$$
(1.41)\quad \frac{\bar{\phi}_{n+1}(0)}{\bar{\phi}_n(0)}z\Theta_n^*(z) - \frac{\kappa_n}{\kappa_{n-1}}\Theta_{n-1}^*(z) = \frac{\phi_{n+1}(0)}{\phi_n(0)}\Theta_n(z) - \frac{\kappa_n}{\kappa_{n-1}}z\Theta_{n-1}(z),
$$

$$
(1.42)\quad \Omega_n^*(z) - \frac{\kappa_{n+1}}{\kappa_n}\Theta_n^*(z) = \Omega_n(z) - \frac{\kappa_{n+1}}{\kappa_n}z\Theta_n(z) + n\frac{W(z)}{z},
$$

$$
(1.43)\quad \Omega_n^*(z) + \Omega_n(z) = \frac{\kappa_n^2}{\kappa_{n+1}^2}\left[\frac{\phi_{n+2}(0)}{\phi_{n+1}(0)}\Theta_{n+1}(z) + \frac{\kappa_{n+1}}{\kappa_n}\Theta_n^*(z)\right] + \frac{W(z)}{z}.
$$

We will require the leading order terms in expansions of $\phi_n(z), \phi_n^*(z), \epsilon_n(z), \epsilon_n^*(z)$ both inside and outside the unit circle. The following Corollary extends the preliminary results reported in [8].

Corollary 1.2 ([8]). *The bi-orthogonal polynomials $\phi_n(z), \phi_n^*(z)$ have the following expansions for $|z| < \delta^- < 1$*

$$\frac{1}{\kappa_n}\phi_n(z) = r_n + [r_{n-1} + r_n\bar{\lambda}_{n-1}]\, z + [r_{n-2} + r_{n-1}\bar{\lambda}_{n-2} + r_n\bar{\mu}_{n-1}]\, z^2 + O(z^3), \tag{1.44}$$

$$\frac{1}{\kappa_n}\phi_n^*(z) = 1 + \bar{\lambda}_n z + \bar{\mu}_n z^2 + \bar{\nu}_n z^3 + O(z^3), \tag{1.45}$$

whilst the associated functions have the expansions

$$\begin{aligned}\frac{\kappa_n}{2}\epsilon_n(z) = z^n - \bar{\lambda}_{n+1}z^{n+1} + [\bar{\lambda}_{n+1}\bar{\lambda}_{n+2} - \bar{\mu}_{n+2}]\, z^{n+2}\\ + [\bar{\lambda}_{n+1}\bar{\mu}_{n+3} + \bar{\lambda}_{n+3}\bar{\mu}_{n+2} - \bar{\nu}_{n+3} - \bar{\lambda}_{n+1}\bar{\lambda}_{n+2}\bar{\lambda}_{n+3}]\, z^{n+3} + O(z^{n+4}),\end{aligned} \tag{1.46}$$

$$\frac{\kappa_n}{2}\epsilon_n^*(z) = \bar{r}_{n+1}z^{n+1} + [\bar{r}_{n+2} - \bar{r}_{n+1}\bar{\lambda}_{n+2}]\, z^{n+2} + [\bar{r}_{n+3} - \bar{r}_{n+2}\bar{\lambda}_{n+3} - \bar{r}_{n+1}\bar{\mu}_{n+3} + \bar{r}_{n+1}\bar{\lambda}_{n+2}\bar{\lambda}_{n+3}]\, z^{n+3} + O(z^{n+4}). \tag{1.47}$$

The large argument expansions $|z| > \delta^+ > 1$ of $\phi_n(z), \phi_n^*(z)$ are

$$\frac{1}{\kappa_n}\phi_n(z) = z^n + \lambda_n z^{n-1} + \mu_n z^{n-2} + \nu_n z^{n-3} + O(z^{n-3}), \tag{1.48}$$

$$\frac{1}{\kappa_n}\phi_n^*(z) = \bar{r}_n z^n + [\bar{r}_{n-1} + \bar{r}_n\lambda_{n-1}]\, z^{n-1} + [\bar{r}_{n-2} + \bar{r}_{n-1}\lambda_{n-2} + \bar{r}_n\mu_{n-1}]\, z^{n-2} + O(z^{n-3}), \tag{1.49}$$

whilst the associated functions have the expansions

$$\frac{\kappa_n}{2}\epsilon_n(z) = r_{n+1}z^{-1} + [r_{n+2} - r_{n+1}\lambda_{n+2}]\, z^{-2} + [r_{n+3} - r_{n+2}\lambda_{n+3} - r_{n+1}\mu_{n+3} + r_{n+1}\lambda_{n+2}\lambda_{n+3}]\, z^{-3} + O(z^{-4}), \tag{1.50}$$

$$\frac{\kappa_n}{2}\epsilon_n^*(z) = 1 - \lambda_{n+1}z^{-1} + [\lambda_{n+2}\lambda_{n+1} - \mu_{n+2}]\, z^{-2} + [\lambda_{n+1}\mu_{n+3} + \lambda_{n+3}\mu_{n+2} - \nu_{n+3} - \lambda_{n+1}\lambda_{n+2}\lambda_{n+3}]\, z^{-3} + O(z^{-4}). \tag{1.51}$$

Proof. Expansions (1.44) and (1.49) are found by differentiating

$$\kappa_n\phi_{n+1}(z) = \kappa_{n+1}z\phi_n(z) + \phi_{n+1}(0)\phi_n^*(z), \tag{1.52}$$

and

$$\phi_{n+1}(0)\bar{\phi}_{n+1}(z) = \kappa_{n+1}z^{n+1}\phi_{n+1}(z^{-1}) - \kappa_n z^n\phi_n(z^{-1}), \tag{1.53}$$

repeatedly, respectively, and setting the argument to zero. Expansion (1.46) can be found using

$$z^n = \frac{\bar{\phi}_n(z)}{\kappa_n} - \bar{\lambda}_n\frac{\bar{\phi}_{n-1}(z)}{\kappa_{n-1}} + [\bar{\lambda}_n\bar{\lambda}_{n-1} - \bar{\mu}_n]\frac{\bar{\phi}_{n-2}(z)}{\kappa_{n-2}} + [\bar{\mu}_n\bar{\lambda}_{n-2} + \bar{\mu}_{n-1}\bar{\lambda}_n - \bar{\nu}_n - \bar{\lambda}_n\bar{\lambda}_{n-1}\bar{\lambda}_{n-2}]\frac{\bar{\phi}_{n-3}(z)}{\kappa_{n-3}} + \Pi_{n-4}. \tag{1.54}$$

Expansion (1.50) is found by making use of

$$z\phi_n(z) = \frac{\kappa_n}{\kappa_{n+1}}\phi_{n+1}(z) - \frac{\phi_{n+1}(0)}{\kappa_n\kappa_{n+1}}\sum_{j=0}^{n}\bar{\phi}_j(0)\phi_j(z), \tag{1.55}$$

repeatedly. Expansion (1.47) can be found using the conjugate analogue of the above equation. □

2. The Regular Semi-classical Class of Weights

Of direct relevance to integrable systems is the regular semi-classical class, characterised by a special structure of their logarithmic derivatives

$$\frac{1}{w(z)}\frac{d}{dz}w(z) = \frac{2V(z)}{W(z)} = \sum_{j=1}^{M}\frac{\rho_j}{z - z_j}, \quad \rho_j \in \mathbb{C}, \tag{2.1}$$

and its degenerate cases. Here $V(z)$, $W(z)$ are polynomials satisfying the following generic conditions for the regular semi-classical class -

(i) deg $(W) \geq 2$,
(ii) deg $(V) <$ deg $(W) = M$,
(iii) the M zeros of $W(z)$, $\{z_1, \ldots, z_M\}$ are distinct, and

(iv) the residues $\rho_j = 2V(z_j)/W'(z_j) \notin \mathbb{Z}_{\geq 0}$.

We have the expansion of the denominator in terms of elementary symmetric functions

$$W(z) = \prod_{j=1}^{M}(z - z_j) = \sum_{l=0}^{M}(-)^l e_l[z_1, \dots, z_M] z^{M-l}, \quad e_0 = 1, \tag{2.2}$$

and of the numerator

$$2V(z) = \sum_{l=0}^{M-1}(-)^l m_l[z_1, \dots, z_M] z^{M-1-l}, \quad m_0 = \sum_{j=1}^{M}\rho_j. \tag{2.3}$$

One explicit example, however not the most general form, of such a weight is the generalised Jacobi weight

$$w(z) = \prod_{j=1}^{M}(z - z_j)^{\rho_j}, \quad \rho_j \in \mathbb{C}, \quad supp(wdz) = \mathbb{T}. \tag{2.4}$$

Lemma 2.1 ([1],[28],[8]). *Let the weight $w(z)$ satisfy the conditions of Proposition 1.3 and $w(e^{2\pi i}) = w(1)$. The Carathéodory function (1.12) satisfies the first order linear ordinary differential equation*

$$W(z)F'(z) = 2V(z)F(z) + U(z), \tag{2.5}$$

where $U(z)$ is a polynomial in z.

Note that we do not assume one of the singularities is located at the origin and the next result is a variant of Proposition 3.1 in [8], which did make that assumption.

Proposition 2.1 ([8]). *For regular semi-classical weights (2.4), the functions $z\Theta_n(z)$, $z\Theta^*_n(z)$, $z\Omega_n(z)$ and $z\Omega^*_n(z)$ in (1.26) are polynomials of degree* $\deg z\Omega_n(z) = \deg z\Omega^*_n(z) = M$, $\deg z\Theta_n(z) = \deg z\Theta^*_n(z) = M - 1$, *independent of n.*

Because of the assumption $z_j \neq 0$ the following result also differs in detail with the corresponding result in [8].

Proposition 2.2 ([8]). *The spectral coefficients have terminating expansions in the interior domain of the unit circle about $z = 0$ with the explicit forms*

$$(-1)^M \frac{\phi_{n+1}(0)}{\phi_n(0)}\Theta_n(z) = -ne_M z^{-1} + \left\{ne_{M-1} - m_{M-1} + e_M\left[(n+1)\bar{\lambda}_{n+1} - (n-1)\left(\bar{\lambda}_{n-1} + \frac{r_{n-1}}{r_n}\right)\right]\right\} + O(z), \tag{2.6}$$

$$(-1)^M \Omega_n(z) = -ne_M z^{-1} + \left\{ne_{M-1} - \frac{1}{2}m_{M-1} + e_M\left[(n+1)\bar{\lambda}_{n+1} - n\left(\bar{\lambda}_n + \frac{r_n}{r_{n+1}}\right)\right]\right\} + O(z), \tag{2.7}$$

$$(-1)^M \frac{\kappa_{n+1}}{\kappa_n}\Theta^*_n(z) = (n+1)e_M z^{-1} + \left\{-(n+1)e_{M-1} + m_{M-1} + e_M\left[(n+2)\left(\frac{\bar{r}_{n+2}}{\bar{r}_{n+1}} - \bar{\lambda}_{n+2}\right) + n\bar{\lambda}_n\right]\right\} + O(z), \tag{2.8}$$

$$(-1)^M \Omega^*_n(z) = (n+1)e_M z^{-1} + \left\{\frac{1}{2}m_{M-1} - (n+1)e_{M-1} + e_M\left[(n+1)\bar{\lambda}_{n+1} + (n+2)\left(\frac{\bar{r}_{n+2}}{\bar{r}_{n+1}} - \bar{\lambda}_{n+2}\right)\right]\right\} + O(z), \tag{2.9}$$

and in the exterior domain of the unit circle about $z = \infty$ with the explicit forms

$$\frac{\kappa_{n+1}}{\kappa_n}\Theta_n(z) = (n+1+m_0)z^{M-2} + \left\{-(n+1)e_1 - m_1 + (n+2+m_0)\left[\frac{r_{n+2}}{r_{n+1}} - \lambda_{n+2}\right] + (n+m_0)\lambda_n\right\}z^{M-3} + O(z^{M-4}), \tag{2.10}$$

$$\Omega_n(z) = (1 + \frac{1}{2}m_0)z^{M-1} + \left\{-e_1 - \frac{1}{2}m_1 + (n+1+m_0)\lambda_{n+1} - (n+2+m_0)\left[\lambda_{n+2} - \frac{r_{n+2}}{r_{n+1}}\right]\right\}z^{M-2} + O(z^{M-3}), \tag{2.11}$$

$$\frac{\bar{\phi}_{n+1}(0)}{\bar{\phi}_n(0)}\Theta^*_n(z) = -(n+m_0)z^{M-2} + \left\{ne_1 + m_1 + (n+1+m_0)\lambda_{n+1} - (n-1+m_0)\left[\lambda_{n-1} + \frac{\bar{r}_{n-1}}{\bar{r}_n}\right]\right\}z^{M-3} + O(z^{M-4}), \tag{2.12}$$

$$\Omega^*_n(z) = -\frac{1}{2}m_0 z^{M-1} + \left\{\frac{1}{2}m_1 + (n+1+m_0)\lambda_{n+1} - (n+m_0)\left[\lambda_n + \frac{\bar{r}_n}{\bar{r}_{n+1}}\right]\right\}z^{M-2} + O(z^{M-3}). \tag{2.13}$$

Proof. These expansions follow from the inversion of (1.26), namely the formulae

$$2\frac{\phi_{n+1}(0)}{\kappa_n}z^n\Theta_n = W\left[-\epsilon'_n\phi_n + \epsilon_n\phi'_n\right] + 2V\epsilon_n\phi_n, \tag{2.14}$$

$$2\frac{\phi_{n+1}(0)}{\kappa_n}z^n\Omega_n = W\left[-\epsilon'_n\phi_{n+1} + \epsilon_{n+1}\phi'_n\right] + V\left[\epsilon_{n+1}\phi_n + \epsilon_n\phi_{n+1}\right], \tag{2.15}$$

$$2\frac{\bar{\phi}_{n+1}(0)}{\kappa_n}z^{n+1}\Theta^*_n = W\left[\epsilon^{*\prime}_n\phi^*_n - \epsilon^*_n\phi^{*\prime}_n\right] - 2V\epsilon^*_n\phi^*_n, \tag{2.16}$$

$$2\frac{\bar{\phi}_{n+1}(0)}{\kappa_n}z^{n+1}\Omega^*_n = W\left[\epsilon^{*\prime}_n\phi^*_{n+1} - \epsilon^*_{n+1}\phi^{*\prime}_n\right] - V\left[\epsilon^*_{n+1}\phi^*_n + \epsilon^*_n\phi^*_{n+1}\right], \tag{2.17}$$

and the expansions of the polynomials and associated functions as given in (1.44-1.51). □

In addition to the coupled equations of Proposition 1.5 and Corollary 1.1, evaluations of the spectral coefficients at the singular points satisfy bilinear relations.

Proposition 2.3 ([8])**.** *For $j = 1, \dots, M$ (i.e. $z_j \neq 0$) the evaluations of the spectral coefficients satisfy the recurrence and functional relations*

$$\Omega^2_n(z_j) = \frac{\kappa_n\phi_{n+2}(0)}{\kappa_{n+1}\phi_{n+1}(0)}z_j\Theta_n(z_j)\Theta_{n+1}(z_j) + V^2(z_j), \tag{2.18}$$

$$\Omega^{*2}_n(z_j) = \frac{\kappa_n\bar{\phi}_{n+2}(0)}{\kappa_{n+1}\bar{\phi}_{n+1}(0)}z_j\Theta^*_n(z_j)\Theta^*_{n+1}(z_j) + V^2(z_j), \tag{2.19}$$

$$\left[\Omega_{n-1}(z_j) - \frac{\kappa^2_{n-1}}{\kappa^2_n}\frac{\phi_{n+1}(0)}{\phi_n(0)}\Theta_n(z_j)\right]^2 = \frac{\phi_{n+1}(0)\bar{\phi}_n(0)}{\kappa^2_n}\Theta_n(z_j)\Theta^*_{n-1}(z_j) + V^2(z_j), \tag{2.20}$$

$$\left[\Omega^*_{n-1}(z_j) - \frac{\kappa^2_{n-1}}{\kappa^2_n}\frac{\bar{\phi}_{n+1}(0)}{\bar{\phi}_n(0)}z_j\Theta^*_n(z_j)\right]^2 = \frac{\kappa_{n-1}\bar{\phi}_{n+1}(0)\phi_n(0)}{\kappa^3_n}z^2_j\Theta^*_n(z_j)\Theta_{n-1}(z_j) + V^2(z_j), \tag{2.21}$$

$$\frac{\phi_{n+1}(0)\bar{\phi}_{n+1}(0)}{\kappa^2_n}z_j\Theta_n(z_j)\Theta^*_n(z_j) = \left[\Omega_n(z_j) + V(z_j) - \frac{\kappa_{n+1}}{\kappa_n}z_j\Theta_n(z_j)\right]\left[\Omega^*_n(z_j) - V(z_j) - \frac{\kappa_{n+1}}{\kappa_n}\Theta^*_n(z_j)\right]. \tag{2.22}$$

It is these relations that lead directly to one of the pair of coupled discrete Painlevé equations.

We note the initial members of the sequences of spectral coefficients $\{\Theta_n\}^\infty_{n=0}$, $\{\Theta^*_n\}^\infty_{n=0}$, $\{\Omega_n\}^\infty_{n=0}$, $\{\Omega^*_n\}^\infty_{n=0}$ are given by

$$2\frac{\phi_1(0)}{\kappa_0}\Theta_0(z) = 2V(z) - \kappa^2_0U(z), \tag{2.23}$$

$$2\frac{\bar{\phi}_1(0)}{\kappa_0}z\Theta^*_0(z) = -2V(z) - \kappa^2_0U(z), \tag{2.24}$$

$$2\phi_1(0)\Omega_0(z) = \kappa_1z[2V(z) - \kappa^2_0U(z)] - \kappa^2_0\phi_1(0)U(z), \tag{2.25}$$

$$2\bar{\phi}_1(0)z\Omega^*_0(z) = -\kappa_1[2V(z) + \kappa^2_0U(z)] - \kappa^2_0\bar{\phi}_1(0)zU(z), \tag{2.26}$$

and observe that $U(z)$ defines the initial values for the recurrences of Proposition 1.5.

Hereafter we restore a singularity of the weight at $z = 0$ (i.e. $\rho_0 \neq 0$) in addition to the previous finite ones so that the degrees of W, V are augmented by a unit (now $M + 1$ and M respectively) and the spectral coefficients Θ_n, Θ^*_n and Ω_n, Ω^*_n are polynomials of degree $M-1$ and M respectively. In this setting the spectral matrix A_n has a partial fraction decomposition

$$A_n = \sum^M_{j=0}\frac{A_{n,j}}{z - z_j}, \quad z_0 = 0, \tag{2.27}$$

which define the residue matrices $A_{n,j}$for $j = 0, \dots, M$. We remark that the spectral matrix has singularities at $z = 0$ and $z = \infty$ irregardless of the locations of singularities of the weight (i.e. zeros of W) due to the bi-orthogonality structure. The j-th residue matrix $A_{n,j}$ at the finite singularity z_j is given by

$$A_{n,j} = \frac{1}{W'(z_j)} \begin{pmatrix} -\Omega_n(z_j) - V(z_j) + \frac{\kappa_{n+1}}{\kappa_n} z_j \Theta_n(z_j) & \frac{\phi_{n+1}(0)}{\kappa_n} \Theta_n(z_j) \\ -\frac{\bar{\phi}_{n+1}(0)}{\kappa_n} z_j \Theta_n^*(z_j) & \Omega_n^*(z_j) - V(z_j) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z_j) \end{pmatrix}, \tag{2.28}$$

with $2V(z_j) = \rho_j W'(z_j)$ for $j = 1, \dots, M$, while for $j = 0$ the expression is

$$A_{n,0} = (n - \rho_0) \begin{pmatrix} 1 & -r_n \\ 0 & 0 \end{pmatrix}, \tag{2.29}$$

and for the singular point $z_{M+1} = \infty$ it is

$$A_{n,\infty} = -\sum_{j=0}^{M} A_{n,j} = \begin{pmatrix} -n & 0 \\ -(n + \sum_{j=0}^{M} \rho_j) \bar{r}_n & \sum_{j=0}^{M} \rho_j \end{pmatrix}. \tag{2.30}$$

In the following section we will draw heavily on partial fraction decompositions of the spectral coefficients which imply the summation identities

$$\sum_{j=0}^{M} \frac{\Theta_n(z_j)}{W'(z_j)} = 0, \tag{2.31}$$

$$\sum_{j=0}^{M} \frac{\Theta_n^*(z_j)}{W'(z_j)} = -(n + \sum_{j=0}^{M} \rho_j) \frac{\bar{\phi}_n(0)}{\bar{\phi}_{n+1}(0)}, \tag{2.32}$$

$$\sum_{j=0}^{M} \frac{\Omega_n(z_j) - V(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j \Theta_n(z_j)}{W'(z_j)} = -(n + \sum_{j=0}^{M} \rho_j), \tag{2.33}$$

$$\sum_{j=0}^{M} \frac{\Omega_n^*(z_j) - V(z_j) - \frac{\kappa_{n+1}}{\kappa_n} \Theta_n^*(z_j)}{W'(z_j)} = -\sum_{j=0}^{M} \rho_j. \tag{2.34}$$

For simplicity we parameterise the free singularities $z_j(t)$ as arbitrary trajectories with respect to a single deformation variable t so that

$$\frac{d}{dt} = \sum_{j=0}^{M} \dot{z}_j \frac{\partial}{\partial z_j}, \tag{2.35}$$

where we include $j = 0$ in the sum for convenience even though $\dot{z}_0 = 0$.

Proposition 2.4 ([8]). *The deformation derivatives of the system (1.13) with respect to arbitrary deformations of the singularities z_j are given by*

$$\frac{d}{dt} Y_n = B_n Y_n = \left[B_\infty - \sum_{j=0}^{M} \dot{z}_j \frac{A_{n,j}}{z - z_j} \right] Y_n, \tag{2.36}$$

where

$$B_\infty = \begin{pmatrix} \frac{\dot{\kappa}_n}{\kappa_n} & 0 \\ \frac{\dot{\kappa}_n \bar{\phi}_n(0) + \kappa_n \dot{\bar{\phi}}_n(0)}{\kappa_n} & -\frac{\dot{\kappa}_n}{\kappa_n} \end{pmatrix}. \tag{2.37}$$

A particularly important deduction from (2.36) are the dynamics of the r-coefficients

$$\frac{\dot{r}_n}{r_n} = \sum_{j=0}^{M} \dot{z}_j \frac{\Omega_{n-1}(z_j) - V(z_j)}{W'(z_j)}, \tag{2.38}$$

$$\frac{\dot{\bar{r}}_n}{\bar{r}_n} = \sum_{j=0}^{M} \dot{z}_j \frac{\Omega^*_{n-1}(z_j) + V(z_j)}{W'(z_j)}. \tag{2.39}$$

Compatibility of the spectral derivative (1.26) and the deformation derivative (2.36) leads to the Schlesinger equations for the residue matrices.

Proposition 2.5 ([8]). *The residue matrices satisfy a system of integrable, non-linear partial differential equations, the Schlesinger equations,*

$$\dot{A}_{n,j} = \left[B_\infty, A_{n,j}\right] + \sum_{\substack{k\neq j\\ 0\leq k\leq M}} \frac{\dot{z}_j - \dot{z}_k}{z_j - z_k}\left[A_{n,k}, A_{n,j}\right], \qquad j = 0,\dots,M, \tag{2.40}$$

$$\dot{A}_{n,\infty} = \left[B_\infty, A_{n,\infty}\right]. \tag{2.41}$$

In the following section we will require a more explicit form for the Schlesinger equations, which were first given in [8].

Lemma 2.2. *The deformation derivatives of the residues of the spectral matrix are given in component form by*

$$\begin{aligned}\frac{\kappa_n}{\phi_{n+1}(0)} W'(z_j)\dot{A}_{nj,12} = {} & \frac{2\dot{\kappa}_n}{\kappa_n}\Theta_n(z_j) \\ & + \sum_{\substack{k\neq j\\ 0\leq k\leq M}} \frac{1}{W'(z_k)}\frac{\dot{z}_j - \dot{z}_k}{z_j - z_k}\Bigg\{\Theta_n(z_k)\left[2\Omega_n(z_j) - 2\frac{\kappa_{n+1}}{\kappa_n} z_j\Theta_n(z_j) + n\frac{W(z_j)}{z_j}\right] \\ & \qquad - \Theta_n(z_j)\left[2\Omega_n(z_k) - 2\frac{\kappa_{n+1}}{\kappa_n} z_k\Theta_n(z_k) + n\frac{W(z_k)}{z_k}\right]\Bigg\},\end{aligned} \tag{2.42}$$

and

$$\begin{aligned} W'(z_j)\dot{A}_{nj,11} = {} & -\frac{\phi_{n+1}(0)}{\kappa_n}\frac{\dot{\kappa}_n\bar{\phi}_n(0) + \kappa_n\dot{\bar{\phi}}_n(0)}{\kappa_n^2}\Theta_n(z_j) \\ & + \sum_{\substack{k\neq j\\ 0\leq k\leq M}} \frac{1}{W'(z_k)}\frac{\dot{z}_j - \dot{z}_k}{z_j - z_k}\Bigg\{\frac{\Theta_n(z_j)}{\Theta_n(z_k)}\left[\Omega_n(z_k) + V(z_k) - \frac{\kappa_{n+1}}{\kappa_n} z_k\Theta_n(z_k)\right]\left[\Omega_n(z_k) - V(z_k) - \frac{\kappa_{n+1}}{\kappa_n} z_k\Theta_n(z_k) + n\frac{W(z_k)}{z_k}\right] \\ & \qquad - \frac{\Theta_n(z_k)}{\Theta_n(z_j)}\left[\Omega_n(z_j) + V(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j\Theta_n(z_j)\right]\left[\Omega_n(z_j) - V(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j\Theta_n(z_j) + n\frac{W(z_j)}{z_j}\right]\Bigg\},\end{aligned} \tag{2.43}$$

for $j = 0,\dots,M$.

Proof. These follow from the Schlesinger equations (2.40) and the transition formulae

$$\Omega^*_n(z_j) - \frac{\kappa_{n+1}}{\kappa_n}\Theta^*_n(z_j) = \Omega_n(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j\Theta_n(z_j) + n\frac{W(z_j)}{z_j}, \tag{2.44}$$

(2.45)
$$\frac{\phi_{n+1}(0)\bar{\phi}_{n+1}(0)}{\kappa_n^2} z_j\Theta^*_n(z_j) = \frac{1}{\Theta_n(z_j)}\left[\Omega_n(z_j) + V(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j\Theta_n(z_j)\right]\left[\Omega_n(z_j) - V(z_j) - \frac{\kappa_{n+1}}{\kappa_n} z_j\Theta_n(z_j) + n\frac{W(z_j)}{z_j}\right],$$

for $j = 0,\dots,M$ and where the ratio $W(z_j)/z_j$ is interpreted as the limit of $z_0 \to 0$ for $j = 0$. □

For each singular point z_j the monodromy matrix M_j is defined by

$$Y_n(z_j + \delta e^{2\pi i}) = Y_n(z_j + \delta)M_j, \tag{2.46}$$

and has the classical, triangular structure

$$M_j = \begin{pmatrix} 1 & c_j(1-e^{-2\pi i\rho_j}) \\ 0 & e^{-2\pi i\rho_j} \end{pmatrix}, \quad j = 0, \dots, M, \tag{2.47}$$

where c_j is independent of the z_j, and thus t, and $n \in \mathbb{Z}_{\geq 0}$ but depends on other details of the weight.

3. The Hamiltonian Formulation and Garnier Systems

We fix the singularities in canonical position (see [20]), i.e. taking them at distinct points

$$z_0 = 0, z_1, \dots, z_N, z_{N+1} = 1, z_{N+2} = \infty, \tag{3.1}$$

with exponents $\rho_0, \rho_1, \dots, \rho_N, \rho_{N+1} = \rho, \rho_{N+2} = \rho_\infty$ respectively, so that the number of finite singularities is $N+2$. Note that now we have a singularity of the weight at the origin, i.e. $\rho_0 \neq 0$. The denominator polynomial for the weight data is given by

$$W(z) = z(z-1)\prod_{j=1}^{N}(z-z_j) = z\sum_{l=0}^{N+1}(-)^{N+1-l}z^l e_{N+1-l}, \tag{3.2}$$

where the elementary symmetric functions of the singularity positions are denoted $e_l, l = 0, \dots, N+1$ and in particular $e_0 = 1$, $e_{N+2} = 0$ and $e_{N+1} = \prod_{j=1}^{N} z_j$. The numerator polynomial is

$$2V(z) = z(z-1)\prod_{j=1}^{N}(z-z_j)\left\{\frac{\rho_0}{z} + \frac{\rho}{z-1} + \sum_{j=1}^{N}\frac{\rho_j}{z-z_j}\right\} = \sum_{l=0}^{N+1}(-)^{N+1-l}z^l m_{N+1-l}, \tag{3.3}$$

where the last relation defines the coefficients $m_l, l = 0, \dots, N+1$ and we observe that $m_0 = \rho_0 + \rho + \sum_{j=1}^{N}\rho_j$ and $m_{N+1} = \rho_0 e_{N+1}$. We can parameterise the upper off-diagonal element of the spectral matrix, i.e. the spectral coefficient Θ_n, which is now of degree N, so that

$$\Theta_n(z) = \Theta_\infty \prod_{r=1}^{N}(z-q_r), \quad \Theta_\infty = (n+1+m_0)\frac{\kappa_n}{\kappa_{n+1}}, \tag{3.4}$$

and thus

$$\frac{\Theta_n'}{\Theta_n} = \sum_{r=1}^{N}\frac{1}{z-q_r}, \tag{3.5}$$

where the poles q_r will play the role of canonical co-ordinates and analogue of the sixth Painlevé transcendent. For notational simplicity we will often suppress the n index dependency as we do not discuss recurrences in this variable in this section. Furthermore we will assume throughout *generic conditions* on the dependent and independent variables, namely that

(i) non-coincidence of singular points, $z_j \neq z_k$ for $j \neq k$ and $j,k = 0, \dots, N+2$, so that $W'(z_j) \neq 0$ for $j = 0, \dots, N+2$
(ii) avoidance by the co-ordinates with each other $q_r \neq q_s$ for $r \neq s$ and $r,s = 1, \dots, N$ implying that $\Theta_n'(q_r) \neq 0$ for $r = 1, \dots, N$ and with the fixed singularities $q_r \neq z_j$ for $r = 1 \dots, N$ and $j = 0, \dots, N+2$ and consequently $W(q_r) \neq 0$ for $r = 1, \dots, N$ and $\Theta_n(z_j) \neq 0$ for $j = 0, \dots, N+2$.

Prior to stating our main result we note some summation identities for $j = 1, \ldots, N$

$$\sum_{\substack{0 \le k \le N+1 \\ k \ne j}} \frac{1}{z_j - z_k} = \frac{1}{2} \frac{W''(z_j)}{W'(z_j)}, \tag{3.6}$$

$$\sum_{\substack{0 \le k \le N+1 \\ k \ne j}} \frac{\rho_k}{z_j - z_k} = \frac{2V'(z_j)}{W'(z_j)} - \frac{V(z_j)W''(z_j)}{[W'(z_j)]^2}, \tag{3.7}$$

$$\sum_{\substack{0 \le k \le N+1 \\ k \ne j}} \frac{\Theta_n(z_k)}{W'(z_k)} \frac{1}{z_j - z_k} = \frac{\Theta'_n(z_j)}{W'(z_j)} - \frac{1}{2} \frac{\Theta_n(z_j)W''(z_j)}{[W'(z_j)]^2}, \tag{3.8}$$

and

$$\sum_{\substack{1 \le s \le N \\ s \ne r}} \frac{1}{q_r - q_s} = \frac{1}{2} \frac{\Theta''_n(q_r)}{\Theta'_n(q_r)}. \tag{3.9}$$

In addition partial fraction expansions imply the following summation identities

$$\sum_{j=0}^{N+1} \frac{\Theta_n(z_j)}{W'(z_j)} z_j^\sigma = \begin{cases} 0, & \sigma = 0 \\ \Theta_\infty, & \sigma = 1 \\ \Theta_\infty[1 + \sum_{k=1}^N z_k - \sum_{r=1}^N q_r], & \sigma = 2 \end{cases}, \tag{3.10}$$

$$\sum_{j=0}^{N+1} \frac{\Theta_n(z_j)}{W'(z_j)} \frac{z_j^\sigma}{z_j - q_r} = \begin{cases} 0, & \sigma = 0, 1 \\ \Theta_\infty, & \sigma = 2 \\ \Theta_\infty[1 + \sum_{k=1}^N z_k - \sum_{s \ne r}^N q_s], & \sigma = 3 \end{cases}, \tag{3.11}$$

for $r = 1, \ldots, N$,

$$\sum_{j=0}^{N+1} \frac{\Theta_n(z_j)}{W'(z_j)} \frac{z_j^\sigma}{(z_j - q_r)(z_j - q_s)} = \begin{cases} -\delta_{s,r} q_r^\sigma \dfrac{\Theta'_n(q_r)}{W(q_r)}, & \sigma = 0, 1, 2 \\ \Theta_\infty - \delta_{s,r} q_r^3 \dfrac{\Theta'_n(q_r)}{W(q_r)}, & \sigma = 3 \\ \Theta_\infty[1 + \sum_{k=1}^N z_k - \sum_{t=1}^N q_t + q_r + q_s] - \delta_{s,r} q_r^4 \dfrac{\Theta'_n(q_r)}{W(q_r)}, & \sigma = 4 \end{cases}, \tag{3.12}$$

for $r, s = 1, \ldots, N$, and

$$\begin{aligned} &\sum_{j=0}^{N+1} \frac{\Theta_n(z_j)}{W'(z_j)} \frac{z_j^\sigma}{(z_j - q_r)(z_j - q_s)(z_j - q_t)} \\ &= -(1 - \delta_{t,s})\delta_{r,s} q_s^\sigma \frac{\Theta'_n(q_s)}{(q_r - q_t)W(q_s)} - (1 - \delta_{s,r})\delta_{t,r} q_r^\sigma \frac{\Theta'_n(q_r)}{(q_t - q_s)W(q_r)} - (1 - \delta_{r,t})\delta_{s,t} q_t^\sigma \frac{\Theta'_n(q_t)}{(q_s - q_r)W(q_t)} \\ &\qquad + \delta_{r,s}\delta_{s,t} q_r^\sigma \frac{\Theta'_n(q_r)}{W(q_r)} \left[\frac{W'(q_r)}{W(q_r)} - \frac{1}{2} \frac{\Theta''_n(q_r)}{\Theta'_n(q_r)} - \frac{\sigma}{q_r} \right], \end{aligned} \tag{3.13}$$

for $r, s, t = 1, \ldots, N$ with $\sigma = 0, 1, 2, 3$ whilst for $\sigma = 4$ an additional term Θ_∞ is necessary.

Proposition 3.1. *Assume that generic conditions apply. The dynamics of the bi-orthogonal system is governed by the Hamiltonian dynamics of the Garnier system $\mathcal{G}_N \equiv \{q_j, p_j; K_j, z_j\}$ with co-ordinate q_r defined above and momenta p_r given by*

$$p_r = -\frac{\Omega_n(q_r) + V(q_r)}{W(q_r)} = A_{1,1}(q_r), \quad r = 1, \ldots, N, \tag{3.14}$$

and the Hamiltonian by

$$K_j = \frac{\Theta_n(z_j)}{W'(z_j)} \sum_{r=1}^N \frac{W(q_r)}{\Theta'_n(q_r)} \frac{1}{z_j - q_r} \left[p_r^2 + p_r \left(\frac{2V(q_r)}{W(q_r)} - \frac{n}{q_r} - \frac{1}{z_j - q_r} \right) - \frac{n(1 + m_0)}{q_r(q_r - 1)} \right]. \tag{3.15}$$

The indicial exponents are given by $\theta_j = -\rho_j$ for $j = 1, \dots, N+1$, $\theta_0 = n - \rho_0$, and $\alpha_\infty = -n$, $\theta_\infty = n + 1 + \sum_{j=0}^{N+1} \rho_j$ with the constant $\kappa = -n(1 + m_0)$. The latter relation is the necessary condition for a classical solution to the Garnier system, and for the "seed" solution of $n = 0$ it vanishes.

Proof. We seek to compare our system with the second order ODE for the Garnier system Eqs. (4.1.1) and (4.1.4) in [20], see also [9], [27],

$$\phi'' + \left\{ \sum_{j=0}^{N+1} \frac{1-\theta_j}{z-z_j} - \sum_{r=1}^{N} \frac{1}{z-q_r} \right\} \phi' + \left\{ \frac{\kappa}{z(z-1)} - \sum_{j=1}^{N} \frac{z_j(z_j-1)K_j}{z(z-1)(z-z_j)} + \sum_{r=1}^{N} \frac{q_r(q_r-1)p_r}{z(z-1)(z-q_r)} \right\} \phi = 0. \tag{3.16}$$

Using Prop. 1.4 we find that

$$p_1 = \frac{\rho_0 + 1 - n}{z} + \frac{\rho + 1}{z-1} + \sum_{j=1}^{N} \frac{\rho_j + 1}{z - z_j} - \sum_{j=1}^{N} \frac{1}{z - q_j}. \tag{3.17}$$

From the residues of p_1 at $z = z_j$ for $j = 0, 1, \dots, N+1$ we can read off the indicial exponents. We also note that

$$p_2 = \frac{1}{z(z-1)} \left[-n(1+m_0) + \mathrm{O}(z^{-1}) \right], \tag{3.18}$$

as $z \to \infty$.

An alternative form of p_2 to (1.30) can be given in terms of the residue matrices

$$p_2 = \sum_{j=0}^{N+1} \sum_{r=1}^{N} \frac{A_{nj,11}}{(z-z_j)(z-q_r)} + \sum_{0 \le j < k \le N+1} \frac{\mathrm{Tr} A_{nj} \mathrm{Tr} A_{nk} - \mathrm{Tr} A_{nj} A_{nk} - A_{nj,11} - A_{nk,11}}{(z-z_j)(z-z_k)}. \tag{3.19}$$

From $p_r = \mathrm{Res}_{z=q_r} p_2(z)$, the above relation and (2.28) we find (3.14). We need to invert this relationship and express $\Omega_n + V$ in terms of the canonical co-ordinate and momenta. Because this variable is a polynomial of degree $N+1$ in the spectral variable z we can use the Lagrange interpolation formula at the nodes $z = 0, q_1, \dots, q_N, 1$ (which are assumed to be distinct)

$$\begin{aligned} \Omega_n + V = \sum_{r=1}^{N} \left[\Omega_n(q_r) + V(q_r) \right] \frac{z(z-1) \prod_{s \ne r}(z-q_s)}{q_r(q_r-1) \prod_{s \ne r}(q_r - q_s)} \\ + \left[\Omega_n(0) + V(0) \right] \frac{(1-z) \prod_s (z-q_s)}{\prod_s(-q_s)} + \left[\Omega_n(1) + V(1) \right] \frac{z \prod_s (z-q_s)}{\prod_s (1-q_s)}. \end{aligned} \tag{3.20}$$

The coefficients of the two last terms are known as $\Omega_n(0) + V(0) = (-)^N (n - \rho_0) e_{N+1}$ from (2.7) and utilising the coefficient of z^{N+1} in (2.11) we deduce

$$\frac{\Omega_n(1) + V(1)}{\Theta_n(1)} = \frac{1 + m_0}{\Theta_\infty} + (-)^N (n - \rho_0) \frac{e_{N+1}}{\Theta_N(0)} + \sum_{r=1}^{N} \frac{p_r W(q_r)}{q_r(q_r-1)\Theta'_n(q_r)}. \tag{3.21}$$

Consequently we conclude that

$$\Omega_n(z) + V(z) - \frac{\kappa_{n+1}}{\kappa_n} z \Theta_n(z) = \Theta_n(z) \left[-\frac{n}{\Theta_\infty} z + (-)^N (n - \rho_0) \frac{e_{N+1}}{\Theta_n(0)} - \sum_{r=1}^{N} \frac{z}{z - q_r} \frac{p_r W(q_r)}{q_r \Theta'_n(q_r)} \right]. \tag{3.22}$$

We also require a similar representation of $2V(z)$ and proceeding in the same manner we find

$$2V(z) = \Theta_n(z) \left[-\rho_0 \frac{W'(0)}{\Theta_n(0)}(z-1) + \rho \frac{W'(1)}{\Theta_n(1)} z + z(z-1) \sum_{r=1}^{N} \frac{1}{z-q_r} \frac{2V(q_r)}{q_r(q_r-1)\Theta'_n(q_r)} \right]. \tag{3.23}$$

From this, under the limit $z \to \infty$, we have the summation

$$\sum_{r=1}^{N} \frac{2V(q_r)}{q_r(q_r-1)\Theta'_n(q_r)} = \frac{m_0}{\Theta_\infty} + \rho_0 \frac{W'(0)}{\Theta_n(0)} - \rho \frac{W'(1)}{\Theta_n(1)}, \tag{3.24}$$

and for $z = z_j$ we have

$$z_j(z_j-1)\sum_{r=1}^{N}\frac{2V(q_r)}{q_r(q_r-1)\Theta_n'(q_r)}\frac{1}{z_j-q_r} = \rho_0\frac{W'(0)}{\Theta_n(0)}(z_j-1) - \rho\frac{W'(1)}{\Theta_n(1)}z_j + \frac{2V(z_j)}{\Theta_n(z_j)}. \tag{3.25}$$

A number of other sums for $j = 1,\dots,N$ which will be required subsequently are

$$\sum_{r=1}^{N}\frac{2V(q_r)}{(z_j-q_r)^2\Theta_n'(q_r)} = \frac{m_0}{\Theta_\infty} - \frac{2V'(z_j)}{\Theta_n(z_j)} + \frac{2V(z_j)\Theta_n'(z_j)}{\Theta_n^2(z_j)}, \tag{3.26}$$

$$\sum_{r=1}^{N}\frac{2V(q_r)}{(z_j-q_r)q_r\Theta_n'(q_r)} = -\frac{m_0}{\Theta_\infty} - \frac{2V'(0)}{z_j\Theta_n(0)} + \frac{2V(z_j)}{z_j\Theta_n(z_j)}, \tag{3.27}$$

and, for $r = 1,\dots,N$ the sum

$$\begin{aligned} q_r(q_r-1)\sum_{s\neq r}^{N}&\frac{2V(q_s)}{q_s(q_s-1)\Theta_n'(q_s)}\frac{1}{q_r-q_s} \\ &= \rho_0\frac{W'(0)}{\Theta_n(0)}(q_r-1) - \rho\frac{W'(1)}{\Theta_n(1)}q_r + \frac{2V(q_r)}{\Theta_n'(q_r)}\left[\frac{V'(q_r)}{V(q_r)} - \frac{1}{2}\frac{\Theta_n''(q_r)}{\Theta_n'(q_r)} - \frac{2q_r-1}{q_r(q_r-1)}\right]. \end{aligned} \tag{3.28}$$

The analogous representation for $W(z)$ takes the form

$$W(z) = z(z-1)\Theta_n(z)\left[\frac{1}{\Theta_\infty} + \sum_{r=1}^{N}\frac{1}{z-q_r}\frac{W(q_r)}{q_r(q_r-1)\Theta_n'(q_r)}\right], \tag{3.29}$$

which enables one to infer the summations

$$\sum_{r=1}^{N}\frac{W(q_r)}{(z_j-q_r)q_r(q_r-1)\Theta_n'(q_r)} = -\frac{1}{\Theta_\infty}, \quad j = 1,\dots,N, \tag{3.30}$$

and for $r = 1,\dots,N$ the sum

$$\sum_{s\neq r}^{N}\frac{W(q_s)}{q_s(q_s-1)\Theta_n'(q_s)}\frac{1}{q_r-q_s} = -\frac{1}{\Theta_\infty} + \frac{W(q_r)}{q_r(q_r-1)\Theta_n'(q_r)}\left[\frac{W'(q_r)}{W(q_r)} - \frac{1}{2}\frac{\Theta_n''(q_r)}{\Theta_n'(q_r)} - \frac{2q_r-1}{q_r(q_r-1)}\right]. \tag{3.31}$$

We proceed in two steps - the first being to compute the derivatives of the canonical variables using the theory from [8] and the second to compute the Hamiltonian itself. Then all we require is a verification of the Hamilton equations of motion.

Our first task is the computation of the deformation derivative of q_r

$$\dot{q}_r = \sum_{k=1}^{N}\dot{z}_k\frac{\partial q}{\partial z_k} = -\mathrm{Res}_{z=q_r}\frac{\dot{A}_{12}}{A_{12}}, \tag{3.32}$$

and where we use the partial fraction expansion (2.27) for the $1,2$ element and the Schlesinger equation (2.42) for $\dot{A}_{nj,12}$. Now we employ the expressions for $\Theta_n(z)$ and $\Omega_n(z) - \frac{\kappa_{n+1}}{\kappa_n}z\Theta_n(z)$ at $z = z_j, z_k$ in terms of the canonical variables as given by (3.4) and (3.22) along with (3.23) in this formula. After interchanging the summation order each term contains a factor which is a sum over the singularity locations and we can evaluate these using the $\sigma = 0$ cases of (3.10), (3.11) and (3.12). Simplifying we find for $r, j = 1,\dots,N$

$$(z_j-q_r)\frac{\partial}{\partial z_j}q_r = \frac{\Theta_n(z_j)W(q_r)}{\Theta_n'(q_r)W'(z_j)}\left[2p_r + \frac{2V(q_r)}{W(q_r)} - \frac{n}{q_r} - \frac{1}{z_j-q_r}\right]. \tag{3.33}$$

We now proceed to the computation of $\dot{p}_r$ which is a more laborious task. We start with the partial fraction expansion for $A_{n,11}(q_r)$ given by (2.27), differentiate with respect to t and employ the $1,1$ component of the Schlesinger equation (2.43) for $\dot{A}_{nj,11}$. Into this expression we must substitute the expressions for $\Theta_n(z)$ and $\Omega_n(z) \pm V(z) - \frac{\kappa_{n+1}}{\kappa_n}z\Theta_n(z)$ at $z = z_j, z_k$ in terms of the canonical variables as given by (3.4) and (3.22) along with (3.23). Again we interchange the summation order which yields in each term a factor of

a singularity sum. To evaluate these sums we employ all of the formulae given in (3.10), (3.11), (3.12) and (3.13) and assemble all the terms together. We also require the formula for $\dot{q}_r$ implied by (3.33). Considerable cancellation occurs at this stage but we are not yet finished. To arrive at the final result we must employ the transcendent sums (3.24) and (3.28), and the result for $r, j = 1, \ldots, N$ is

$$
\begin{aligned}
(z_j - q_r)\frac{W'(z_j)}{\Theta_n(z_j)}\frac{\partial}{\partial z_j}p_r = -\frac{W(q_r)}{\Theta'_n(q_r)}\Bigg[& p_r^2\left(\frac{W'(q_r)}{W(q_r)} - \frac{1}{2}\frac{\Theta''_n(q_r)}{\Theta'_n(q_r)}\right) + p_r\frac{2V(q_r)}{W(q_r)}\left(\frac{V'(q_r)}{V(q_r)} - \frac{1}{2}\frac{\Theta''_n(q_r)}{\Theta'_n(q_r)}\right) \\
& -n\frac{p_r}{q_r}\left(\frac{W'(q_r)}{W(q_r)} - \frac{1}{2}\frac{\Theta''_n(q_r)}{\Theta'_n(q_r)} - \frac{1}{q_r}\right) - \frac{p_r}{z_j - q_r}\left(\frac{W'(q_r)}{W(q_r)} - \frac{1}{2}\frac{\Theta''_n(q_r)}{\Theta'_n(q_r)} + \frac{1}{z_j - q_r}\right) + n(1+m_0)\frac{1}{q_r(q_r-1)(z_j-q_r)}\Bigg] \\
& - \sum_{s\neq r}\frac{W(q_s)}{\Theta'_n(q_s)}\frac{1}{q_s - q_r}\left[p_s^2 + p_s\frac{2V(q_s)}{W(q_s)} - n\frac{p_s}{q_s} - \frac{p_s}{z_j - q_s} + n(1+m_0)\frac{q_s - q_r}{q_s(q_s-1)(z_j-q_s)}\right].
\end{aligned}
\tag{3.34}
$$

Our starting point for the computation of the Hamiltonian is the formula

$$
K_j = -A_{nj,11}\sum_{r=1}^{N}\frac{1}{z - q_r} - \sum_{\substack{0\leq k\leq N+1 \\ k\neq j}}\frac{1}{z_j - z_k}\left[\mathrm{Tr}A_{nj}\mathrm{Tr}A_{nk} - \mathrm{Tr}A_{nj}A_{nk} - A_{nj,11} - A_{nk,11}\right], \quad j = 1, \ldots, N. \tag{3.35}
$$

We substitute (2.28) and (2.29) for the residues of the spectral matrix in the above formula in conjunction with the transition relations (2.44) and (2.45). Into the resulting expression we must further substitute the representations of the spectral coefficients (3.4), (3.22) and (3.23). Some care needs to be exercised to ensure that the $j, k = 0$ contributions are correctly accounted for and this leads to a separation between these terms and the remaining generic ones. In the group of generic terms there is a summation over z_k for $0 \leq k \leq N+1$ and in these terms we can employ the evaluations given in (3.6), (3.7), (3.8), (3.10), (3.11) and (3.12). After reinstating these terms in the total expression considerable cancellation is effected. Still further simplification is possible by using the transcendent sums (3.24), (3.26), (3.27), (3.25) and lastly (3.30). The final result yields (3.15) which is precisely Eq.(4.3.7) in [20] after making all the notational correspondences. The verification of (3.33) and (3.34) using this Hamiltonian is a straight forward calculation. $\square$

Remark 3.1. The Riemann-Papperitz symbol (see subsection 1.4 of [20], "Fuchsian Equations" for the definition) for our system is

$$
\left\{\begin{array}{cccccccccc}
z_0 = 0 & z_1 & \cdots & z_N & z_{N+1} = 1 & z_{N+2} = \infty & q_1 & \cdots & q_N \\
0 & 0 & \cdots & 0 & 0 & -n & 0 & \cdots & 0 \\
n - \rho_0 & -\rho_1 & \cdots & -\rho_N & -\rho & 1 + \sum_{j=0}^{N+1}\rho_j & 2 & \cdots & 2
\end{array}\right\}. \tag{3.36}
$$

Remark 3.2. One could formulate the Hamiltonian system using the "conjugate" set of variables Θ^*_n, Ω^*_n instead, and in fact all of the dynamical description, however we do not carry out this task as no new understanding would result from it.

Remark 3.3. This Hamiltonian system is not polynomial in the canonical variables q_r and the dynamical equations for q_r do not possess the Painlevé property in z_j, however using the well-known canonical transformation to the new Hamiltonian system $\mathcal{H}_N = \{Q_j, P_j; H_j, t_j\}$ [27], [20]

$$
t_j = \frac{z_j}{z_j - 1}, \tag{3.37}
$$

$$
Q_j = \frac{z_j\Theta_n(z_j)}{\Theta_\infty W'(z_j)}, \tag{3.38}
$$

$$
P_j = -(z_j - 1)\sum_{r=1}^{N}\frac{p_r}{q_r(q_r-1)}\frac{\Theta_\infty W(q_r)}{(q_r - z_j)\Theta'_n(q_r)}, \tag{3.39}
$$

both these deficiencies can be removed.

Remark 3.4. The definitions of the co-ordinates (3.4) and (3.14), (3.15) bear some resemblance to those in Syklanin's separation of variables procedure [15], [16] although we cannot offer any explanation of this here.

4. The discrete Garnier Equations for the $L(1^{M+1};2)$ Garnier Systems

In this section we derive recurrences in n or equivalently when $\theta_0 \mapsto \theta_0 \pm 1$ and $\theta_\infty \mapsto \theta_\infty \pm 1$ for the appropriate variables, thereby characterising the system in this way. We will find the discrete fifth Painlevé equation as the simplest case corresponding to one free deformation variable and the higher analogues of this recurrence system for the multi-variable generalisations. These are the discrete Garnier equations.

The system with $M = 3, N = 1$ singularities at the standard positions

$$\left\{ \begin{array}{cccc} 0 & t & 1 & \infty \\ n-\rho_0 & -\rho_t & -\rho_1 & n+\rho_0+\rho_t+\rho_1 \end{array} \right\}, \tag{4.1}$$

corresponds to sixth Painlevé isomonodromic system and was treated extensively in [7], especially with regard to the forms of the discrete Painlevé equations. The weight data is

$$W(z) = z(z-t)(z-1) = z^3 - e_1 z^2 + e_2 z - e_3, \quad 2V(z) = W \sum_{j=0,t,1} \frac{\rho_j}{z-z_j} = m_0 z^2 - m_1 z + m_2. \tag{4.2}$$

The spectral coefficients can be parameterised in the following way

$$\frac{\kappa_{n+1}}{\kappa_n}\Theta_n(z) = \vartheta_n + (n+1+m_0)z, \tag{4.3}$$

$$\Omega_n(z) = -ne_2 + \frac{1}{2}m_2 - (\omega_n - \frac{1}{2}m_1)z + (1+\frac{1}{2}m_0)z^2, \tag{4.4}$$

whilst the sub-leading coefficients can be related to the polynomial coefficients themselves by

$$\vartheta_n = -\frac{r_n}{r_{n+1}}(n-\rho_0)t, \tag{4.5}$$

$$\omega_n = 1+\rho_0+\rho_t+(1+\rho_0+\rho_1)t + (n+2+m_0)\left[\lambda_{n+2} - \frac{r_{n+2}}{r_{n+1}}\right] - (n+1+m_0)\lambda_{n+1}. \tag{4.6}$$

Proposition 4.1 ([7]). *The n-recurrence for the bi-orthogonal polynomial system with the $M = 3$ regular semi-classical weight is governed by the system of coupled first order difference equations*

$$tf_n f_{n+1} = \frac{[\omega_n + n - t - \rho_0(t+1) - (\rho_t+\rho_1)t][\omega_n + n - t - \rho_0(t+1) - \rho_t - \rho_1 t]}{[\omega_n + nt - 1 - \rho_0(t+1) - \rho_t - \rho_1][\omega_n + nt - 1 - \rho_0(t+1) - \rho_t - \rho_1 t]}, \tag{4.7}$$

and

$$\omega_n + \omega_{n-1} + (2n-1)t - 2 - 2\rho_0(t+1) - 2\rho_t - \rho_1(t+1) = (n-\rho_0)\frac{1-t}{f_n - 1} + (n+1+m_0)\frac{1-t}{tf_n - 1}. \tag{4.8}$$

The transformations relating these variables to the bi-orthogonal system are given by

$$tf_n := \frac{\Theta_n(t)}{\Theta_n(1)}. \tag{4.9}$$

Proof. We refer the reader to the proof of the next case, Proposition 4.2, as the methods employed in both cases are the same. □

The above coupled recurrence system is equivalent to the canonical "discrete fifth Painlevé equation" [13], [30] with the mapping

$$t \mapsto 1/t, \quad \omega_n \mapsto (1-t)\omega_n - nt + 1 + \rho_0(t+1) + \rho_t + \rho_1, \tag{4.10}$$

and the identification

$$\alpha_0 = \rho_t, \quad \alpha_1 = n-\rho_0, \quad \alpha_2 = -n-\rho_t-\rho_1, \quad \alpha_3 = \rho_1, \quad \alpha_4 = n+1+\rho_0+\rho_t+\rho_1. \tag{4.11}$$

In the case $M = 4, N = 2$ we have the two-variable generalisation of the sixth Painlevé equation or the $L(1^5;2)$ two-variable Garnier system. A standard placement of the singularities, without loss of generality, would be

$$\left\{ \begin{array}{ccccc} 0 & s & t & 1 & \infty \\ n-\rho_0 & -\rho_s & -\rho_t & -\rho_1 & n+\rho_0+\rho_s+\rho_t+\rho_1 \end{array} \right\}. \tag{4.12}$$

We write for notational convenience the weight data in the following way

$$W(z) = z(z-1)(z-s)(z-t) = z^4 - e_1 z^3 + e_2 z^2 - e_3 z + e_4, \tag{4.13}$$

$$2V(z) = W \sum_{j=0,s,t,1} \frac{\rho_j}{z-z_j} = m_0 z^3 - m_1 z^2 + m_2 z - m_3. \tag{4.14}$$

The Toeplitz elements satisfy the third order linear difference equation

$$\begin{aligned}(j-\rho_0)stw_j - \left[(j-1-\rho_0)(s+t+st) - \rho_1 st - \rho_s t - \rho_t s\right] w_{j-1} \\ + \left[(j-2-\rho_0)(1+s+t) - \rho_1(s+t) - \rho_s(t+1) - \rho_t(s+1)\right] w_{j-2} \\ - (j-3-\rho_0-\rho_1-\rho_s-\rho_t) w_{j-3} = 0, \quad j \in \mathbb{Z},\end{aligned} \tag{4.15}$$

and we take w_{-1}, w_0, w_1 as defining a solution of the system. Also, according to (2.10)-(2.7), we can parameterise the spectral coefficients as

$$\frac{\kappa_{n+1}}{\kappa_n}\Theta_n(z) = (n-\rho_0)e_3\frac{r_n}{r_{n+1}} + \vartheta_n z + (n+1+m_0)z^2, \tag{4.16}$$

$$\Omega_n(z) = ne_3 - \frac{1}{2}m_3 + (\omega_n - \frac{1}{2}m_2)z + (\varpi_n + \frac{1}{2}m_1)z^2 + (1+\frac{1}{2}m_0)z^3, \tag{4.17}$$

in terms of ϑ_n, ω_n and ϖ_n.

We can relate the new variables introduced above to the coefficients of the polynomials through the explicit forms of the spectral coefficients (2.10)-(2.7) in the following way

$$\vartheta_n = -(n+1)e_1 - m_1 + (n+2+m_0)\left[\frac{r_{n+2}}{r_{n+1}} - r_{n+2}\bar{r}_{n+1}\right] - (n+m_0)r_{n+1}\bar{r}_n - 2\lambda_{n+1}, \tag{4.18}$$

$$\omega_n = -ne_2 + m_2 + (n-\rho_0)e_3\left[\frac{r_n}{r_{n+1}} - \bar{r}_{n+1}r_n\right] - e_3\bar{\lambda}_{n+1}, \tag{4.19}$$

$$\varpi_n = -e_1 - m_1 + (n+2+m_0)\left[\frac{r_{n+2}}{r_{n+1}} - r_{n+2}\bar{r}_{n+1}\right] - \lambda_{n+1}. \tag{4.20}$$

These formulae only serve to allow the recovery of the original variables and do not feature in the recurrence relations.

At this point it is possible to use the foregoing results to derive a system of recurrence relations which is the analogue of the fifth discrete Painlevé equation for the two-variable Garnier system.

Proposition 4.2. *The following system of coupled first order recurrence relations in n for the variables* $\{f_n, g_n, \omega_n, \varpi_n\}_{n=0}^{\infty}$ *completely characterises the bi-orthogonal polynomial system*

$$sf_nf_{n+1} = \frac{\left[\omega_n + s\varpi_n + (1+m_0)s^2 + (n-\rho_0)t + \rho_s(s-t)(1-s)\right]\left[\omega_n + s\varpi_n + (1+m_0)s^2 + (n-\rho_0)t\right]}{\left[\omega_n + \varpi_n + 1 + m_0 + (n-\rho_0)st + \rho_1(1-s)(t-1)\right]\left[\omega_n + \varpi_n + 1 + m_0 + (n-\rho_0)st\right]}, \tag{4.21}$$

$$tg_ng_{n+1} = \frac{\left[\omega_n + t\varpi_n + (1+m_0)t^2 + (n-\rho_0)s + \rho_t(t-s)(1-t)\right]\left[\omega_n + t\varpi_n + (1+m_0)t^2 + (n-\rho_0)s\right]}{\left[\omega_n + \varpi_n + 1 + m_0 + (n-\rho_0)st + \rho_1(1-s)(t-1)\right]\left[\omega_n + \varpi_n + 1 + m_0 + (n-\rho_0)st\right]}, \tag{4.22}$$

$$\begin{aligned}\omega_n + \omega_{n-1} = m_2 - (n-1)(s+t+st) \\ + (n-\rho_0)\frac{s^2-t^2+(1-s^2)tg_n-(1-t^2)sf_n}{t-s+(1-t)f_n-(1-s)g_n} + (n+1+m_0)st\frac{t-s+(1-t)f_n-(1-s)g_n}{t-s+(1-t)sf_n-(1-s)tg_n},\end{aligned} \tag{4.23}$$

$$\begin{aligned}\varpi_n + \varpi_{n-1} = -m_1 + (n-1)(1+s+t) \\ + (n-\rho_0)\frac{t-s+(1-t)sf_n-(1-s)tg_n}{t-s+(1-t)f_n-(1-s)g_n} + (n+1+m_0)\frac{s^2-t^2+(1-s^2)tg_n-(1-t^2)sf_n}{t-s+(1-t)sf_n-(1-s)tg_n},\end{aligned} \tag{4.24}$$

where

$$sf_n := \frac{\Theta_n(s)}{\Theta_n(1)}, \quad tg_n := \frac{\Theta_n(t)}{\Theta_n(1)}. \tag{4.25}$$

The recurrence relations are subject to the initial conditions

$$f_0 = \frac{(1+m_0)sw_{-1} - [\rho_0 s(t+1) + \rho_1 st + \rho_s t + \rho_t s]\, w_0 - (1-\rho_0)stw_1}{(1+m_0)w_{-1} - [\rho_0(s+t) + \rho_1 st + \rho_s t + \rho_t s]\, w_0 - (1-\rho_0)stw_1}, \tag{4.26}$$

$$g_0 = \frac{(1+m_0)tw_{-1} - [\rho_0 t(s+1) + \rho_1 st + \rho_s t + \rho_t s]\, w_0 - (1-\rho_0)stw_1}{(1+m_0)w_{-1} - [\rho_0(s+t) + \rho_1 st + \rho_s t + \rho_t s]\, w_0 - (1-\rho_0)stw_1}, \tag{4.27}$$

$$\omega_0 = (1-\rho_0)st\frac{w_1}{w_0} + \rho_0 st\frac{w_0}{w_{-1}} + \rho_0(s+t+st) + \rho_1 st + \rho_s t + \rho_t s, \tag{4.28}$$

$$\varpi_0 = -(1+m_0)\frac{w_{-1}}{w_0} - (1-\rho_0)st\frac{w_1}{w_{-1}} - [\rho_0(s+t+st) + \rho_1 st + \rho_s t + \rho_t s]\,\frac{w_0}{w_{-1}}. \tag{4.29}$$

Proof. The first two relations (4.21,4.22) follow from the evaluation of (2.18) at $j = s, t, 1$ and taking ratios. From the definitions (4.25) and (4.16) we note that

$$f_n = \frac{(n-\rho_0)t\frac{r_n}{r_{n+1}} + \vartheta_n + (n+1+m_0)s}{(n-\rho_0)st\frac{r_n}{r_{n+1}} + \vartheta_n + n + 1 + m_0}, \tag{4.30}$$

$$g_n = \frac{(n-\rho_0)s\frac{r_n}{r_{n+1}} + \vartheta_n + (n+1+m_0)t}{(n-\rho_0)st\frac{r_n}{r_{n+1}} + \vartheta_n + n + 1 + m_0}. \tag{4.31}$$

Solving for ϑ_n and r_n/r_{n+1} in terms of f_n, g_n we find that

$$\vartheta_n = (n+1+m_0)\frac{s^2 - t^2 + (1-s^2)tg_n - (1-t^2)sf_n}{t - s + (1-t)sf_n - (1-s)tg_n}, \tag{4.32}$$

and

$$\frac{r_n}{r_{n+1}} = \frac{n+1+m_0}{n-\rho_0}\,\frac{t - s + (1-t)f_n - (1-s)g_n}{t - s + (1-t)sf_n - (1-s)tg_n}. \tag{4.33}$$

The second pair of relations (4.23,4.24) follow from the evaluation of the recurrence (1.33) at $z = s, t$, then solving for their left-hand sides and utilising the preceding expressions. □

We now give the result for an arbitrary number of variables in a canonical placement expressed by the abbreviated symbol

$$\left\{ \begin{array}{cccccc} 0 & t_1 & \cdots & t_N & 1 & \infty \\ n-\rho_0 & -\rho_1 & \cdots & -\rho_N & -\rho & n + \sum_{j=0}^{N+1} \rho_j \end{array} \right\}, \tag{4.34}$$

with $M = N + 2$. To begin with we denote the l-th elementary symmetric function in the variables $t_1, \dots, t_n$ by $e_l(t_1, \dots, t_n)$ and the convention $e_0 = 1$. We adopt the definition and notations for W and $2V$ as given by (3.2) and (3.3) respectively, along with this renaming of the independent variables. Let us define the set $T := \{t_1, \dots, t_N\}$ and the set omitting the variable t_j by $T_j := T\backslash\{t_j\}$. Furthermore we define the Vandermonde determinant

$$\Delta(T) := \prod_{1\le j<k\le N} (t_k - t_j). \tag{4.35}$$

The defining relation for the weight (2.1) implies that the Toeplitz elements $\{w_k\}_{k=-\infty}^{\infty}$ satisfy the N+1-order linear difference equation

$$\sum_{l=0}^{N+1} (-)^l\,[(j-l)e_{N+1-l} - m_{N+1-l}]\, w_{j-l} = 0, \quad j \in \mathbb{Z}, \tag{4.36}$$

with $N+1$ consecutive elements being arbitrary "initial" values. These "initial" values also define the U polynomial, which will in turn fix the initial values of our recurrence relations 4.3, by the expression

$$U := \sum_{l=0}^{N+1} u_l z^l, \tag{4.37}$$

where the coefficients are

$$u_0 = (-)^N w_0 m_{N+1}, \quad u_{N+1} = w_0 m_0, \tag{4.38}$$

$$u_j = (-)^{N-j} w_0 m_{N+1-j} + 2\sum_{l=0}^{j-1} (-)^{N+1-l} \left[(j-l)e_{N+1-l} - m_{N+1-l}\right] w_{j-l}, \quad j = 1, \dots, N. \tag{4.39}$$

We adopt a parameterisation of the spectral coefficients of the form

$$\frac{\kappa_{n+1}}{\kappa_n}\Theta_n(z) = (-)^N(ne_{N+1} - m_{N+1})\frac{r_n}{r_{n+1}} + \sum_{l=1}^{N-1} \vartheta_n^l z^l + (n+1+m_0)z^N, \tag{4.40}$$

and

$$\Omega_n(z) = (-)^N(ne_{N+1} - \frac{1}{2}m_{N+1}) + (-)^N \sum_{l=1}^{N} (\omega_n^l + \frac{1}{2}(-)^l m_{N+1-l}) z^l + (1 + \frac{1}{2}m_0) z^{N+1}, \tag{4.41}$$

which introduces the variables $\vartheta_n^1, \dots, \vartheta_n^{N-1}$ and $\omega_n^1, \dots, \omega_n^N$.

Proposition 4.3. *Define the variables*

$$t_j f_n^j := \frac{\Theta_n(t_j)}{\Theta_n(1)}, \quad j = 1, \dots, N. \tag{4.42}$$

The following system of $2N$ *coupled first order recurrence relations in* n *for the variables* $\{f_n^j, \omega_n^j\}_{j=1}^N$

$$t_j f_n^j f_{n+1}^j = \frac{\left[(n-\rho_0)\prod_{k\neq j} t_k + \sum_{l=1}^N t_j^{l-1}\omega_n^l + (-)^N(1+m_0)t_j^N\right]}{\left[(n-\rho_0)\prod_k t_k + \sum_{l=1}^N \omega_n^l + (-)^N(1+m_0)\right]} \times \frac{\left[n\prod_{k\neq j} t_k + \sum_{l=1}^N t_j^{l-1}(\omega_n^l + (-)^l m_{N+1-l}) + (-)^N t_j^N\right]}{\left[n\prod_k t_k + \sum_{l=1}^N(\omega_n^l + (-)^l m_{N+1-l}) + (-)^N\right]}, \quad j = 1, \dots, N, \tag{4.43}$$

and

$$\begin{aligned} \omega_n^j + \omega_{n-1}^j + (-)^j m_{N+1-j} &= (-)^j(n-1)e_{N+1-j}(T\cup\{1\}) \\ &+ (-)^j(n-\rho_0)\frac{\Delta(T)e_{N-j}(T) + \sum_{l=1}^N (-)^{N-1+l} t_l \Delta(T_l\cup\{1\})e_{N-j}(T_l\cup\{1\})f_n^l}{\Delta(T) + \sum_{l=1}^N (-)^{N-1+l}\Delta(T_l\cup\{1\})f_n^l} \\ &- (-)^j(n+1+m_0)\frac{\Delta(T)e_{N+1-j}(T) + \sum_{l=1}^N (-)^{N-1+l} t_l \Delta(T_l\cup\{1\})e_{N+1-j}(T_l\cup\{1\})f_n^l}{\Delta(T) + \sum_{l=1}^N (-)^{N-1+l} t_l \Delta(T_l\cup\{1\})f_n^l}, \quad j = 1, \dots, N, \end{aligned} \tag{4.44}$$

completely characterises the bi-orthogonal polynomial system. The recurrence relations are subject to the initial conditions

$$f_0^j = \frac{2V(t_j) - \kappa_0^2 U(t_j)}{t_j[2V(1) - \kappa_0^2 U(1)]}, \tag{4.45}$$

$$(-)^N \omega_0^j = -\frac{1}{2}[z^j]\left(2V + \kappa_0^2 U\right) - \frac{w_0}{2w_{-1}}[z^{j-1}]\left(2V - \kappa_0^2 U\right), \tag{4.46}$$

for $j = 1, \dots, N$, *where* $[z^l](.)$ *denotes the coefficient of* z^l *in the polynomial.*

Proof. To establish (4.43) we utilise (2.18) in the following form

$$\frac{t_j\Theta_n(t_j)\Theta_{n+1}(t_j)}{\Theta_n(1)\Theta_{n+1}(1)} = \frac{[\Omega_n(t_j) + V(t_j)][\Omega_n(t_j) - V(t_j)]}{[\Omega_n(1) + V(1)][\Omega_n(1) - V(1)]}, \quad j = 1, \dots, N. \tag{4.47}$$

We note that the factors appearing on the right-hand side can be found from

$$\Omega_n + V = (-)^N(n-\rho_0)e_{N+1} + (-)^N\sum_{l=1}^{N}\omega_n^l z^l + (1+m_0)z^{N+1}, \tag{4.48}$$

$$\Omega_n - V = (-)^N n e_{N+1} + (-)^N\sum_{l=1}^{N}(\omega_n^l + (-)^l m_{N+1-l})z^l + z^{N+1}. \tag{4.49}$$

Then (4.43) follows from this result and the definition (4.42).

To prove the relations (4.44) we first need to invert the definition (4.42) along with the parameterisation (4.40) for the coefficients r_n/r_{n+1} and ϑ_n^j for $j = 1,\dots,N$. We find

$$(n-\rho_0)\frac{r_n}{r_{n+1}} = (n+1+m_0)\frac{\Delta(T) + \sum_{l=1}^{N}(-)^{N-1+l}\Delta(T_l\cup\{1\})f_n^l}{\Delta(T) + \sum_{l=1}^{N}(-)^{N-1+l}t_l\Delta(T_l\cup\{1\})f_n^l}, \tag{4.50}$$

and

$$\vartheta_n^j = (-)^{N+j}(n+1+m_0)\frac{\Delta(T)e_{N-j}(T) + \sum_{l=1}^{N}(-)^{N-1+l}t_l\Delta(T_l\cup\{1\})e_{N-j}(T_l\cup\{1\})f_n^l}{\Delta(T) + \sum_{l=1}^{N}(-)^{N-1+l}t_l\Delta(T_l\cup\{1\})f_n^l}, \tag{4.51}$$

for $j = 1,\dots,N-1$ by using the identity for Vandermonde determinants $\Delta_j(T)$ with the j-th column missing

$$\Delta_j(T) := \det\begin{pmatrix} 1 & t_1 & \cdots & [] & \cdots & t_1^N \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & t_N & \cdots & [] & \cdots & t_N^N \end{pmatrix} = e_{N-j}(T)\Delta_N(T), \tag{4.52}$$

for $j = 0,\dots,N$. The initial values of the dependent variables (4.45 and (4.46) are found using (2.23) and (2.25) respectively and their definitions. □

Remark 4.1. Of primary interest in applications are the Toeplitz determinants (1.2) which are also τ-functions in the sense of Jimbo-Miwa-Ueno's definition [23],[21],[22]. Recovery of the τ-function I_n from the f_n^j, ω_n^j of the previous proposition can be achieved by solving the recurrence in r_n using (4.50) and then by employing either of

$$\vartheta_n^{N-1} = -(n+1)e_1 - m_1 + (n+2+m_0)\left(\frac{r_{n+2}}{r_{n+1}} - \lambda_{n+2}\right) + (n+m_0)\lambda_n, \tag{4.53}$$

or

$$(-)^N\omega_n^N = -e_1 - m_1 + (n+1+m_0)\lambda_{n+1} + (n+2+m_0)\left(\frac{r_{n+2}}{r_{n+1}} - \lambda_{n+2}\right), \tag{4.54}$$

and solving these recurrences for λ_n. This latter sequence can then used to find $\bar{r}_n$ via (1.9), and with both of the r-coefficients one can use (1.10) to solve for the recurrence in I_n.

Remark 4.2. The mapping between the Hamiltonian variables for $\mathcal{G}_N$ and those of the above proposition are given by (4.42) with (3.4) and

$$-W(q_r)p_r = (-)^N(n-\rho_0)e_{N+1} + (-1)^N\sum_{j=1}^{N}\omega_n^j q_r^j + (1+m_0)q_r^{N+1}, \tag{4.55}$$

or

$$(-)^j\omega_n^j = \left[1+m_0+(n-\rho_0)\frac{e_N(T)}{e_N(\mathcal{Q})}\right]e_{N-j}(\mathcal{Q}) - \sum_{r=1}^{N}e_{N-j}(\mathcal{Q}_r)\frac{(q_r-1)\prod_{k=1}^{N}(q_r-t_k)}{\prod_{s\neq r}(q_r-q_s)}p_r, \tag{4.56}$$

where $\mathcal{Q} := \{q_s\}_{s=1}^N$ and $\mathcal{Q}_r := \mathcal{Q}\backslash\{q_r\}$. Written in these Hamiltonian co-ordinates the second of the coupled recurrences (4.44) is

$$p_{n+1,r} + p_{n,r} = \frac{n}{q_r} - \frac{2V(q_r)}{W(q_r)}. \tag{4.57}$$

Remark 4.3. A system of discrete Garnier equations could be formulated using the "conjugate" variables Θ_n^* and Ω_n^* but this would not differ in essence from the one presented here.

Remark 4.4. We have not attempted to check whether singularity confinement [12], an algebraic entropy criteria [18], [4], [35] or one based on Nevalinna theory [2], [14] applies to our recurrences and this remains an outstanding issue.

Remark 4.5. There have been reports of recurrence relations for the Garnier systems in [36] and [31], however the explicit relationship between these equations and the recurrences reported here remains to be elucidated.

Remark 4.6. Another possible method for deriving the recurrence relations of Proposition 4.3 would be to construct the Schlesinger transformation, or lattice translation, operators from the fundamental reflection and automorphism operators of the affine Weyl group $B^{(1)}_{N+3}$, which have been studied in [26], [37], [33]. This task was carried out for $M = 3, N = 1$, i.e. for the sixth Painlevé equation in [8], however this involved a laborious calculation and it may not be feasible to employ this approach to the multi-variable extension.

In this study we have focused on a particular type of transformation, namely that of $n \mapsto n \pm 1$ or equivalently $\theta_0 \mapsto \theta_0 \pm 1$ and $\theta_\infty \mapsto \theta_\infty \pm 1$, which is natural within this context. However one could legitimately ask for the recurrence systems for the transformations $\rho_j \mapsto \rho_j \pm 1$, which are part of the larger group of Schlesinger transformations. The theory of these, in the context of bi-orthogonal system on the unit circle, has been investigated in [38] but analogues of the recurrences found here were not given there.

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